Thinking via Pictures: Getting Students Started through Graphing

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Abstract

Visual languages are among the most important to the STEM disciplines, but most students entering a STEM curriculum seem to have little ability to think or converse in any visual language. Further, there are few curricula that include a formal course in either basic graphics or the art of approximation. One foundational visual language is that of two-dimensional presentation of quantitative information and mathematical relationships. This paper offers some topics to consider for those interested in helping students (1) build basic skills in producing graphs quickly and (2) gain experience in thinking about and understanding relationships, using graphs as aids.

Introduction

This paper is simply a call to acquaint students in science, technology, engineering or mathematics (STEM) with the skills involved in the display of quantitative information. For the typical STEM undergraduate, whether freshman or advanced, the art of thinking graphically seems largely nonexistent. There are certainly some graphing aids available. Essentially every college freshman owns a graphing calculator. They were required to use them in their middle-school and high-school math courses. But how many of those calculators provide enough resolution to read a value from the curve to within, say, five percent? How easy is it to plot multiple curves on the same set of axes? Or plot asymptotes? Or draw curves described in general, symbolic terms? Or produce easily, say, a Bode plot or a root-locus plot? Or enable quick approximation? Etc. The combination of a computer, a good graphing package, and a programming language or a set of appropriate applications can accomplish the above, but such comes at a price—in monetary outlay, and in time required to gain familiarity with all the elements of the combination. And neither approach, whether via calculator or computer, exercises significantly the hand–brain connection, which, for most, is so vital to developing understanding.

There are a vast number of ways to think using pencil and paper; and, for a scientist or engineer, being able to sketch mathematical functions, relationships, and approximations quickly—and, if necessary, without the aid of a calculator or computer—is vital to efficient first-order analysis and preliminary design. Seasoned scientists and engineers don't refer to "back-of-the-envelope" calculations for no good reason. Indeed, there are too many stories floating around of gurus of analysis and design and their amazing "back-of-the-envelope" sketches and calculations to all be apocryphal.

What do we do to help our students discover the utility and pleasure of thinking visually, and how do we help them develop a reasonable agility with the art? We need to not only present results graphically, but also assist the students by having them slow down long enough to think, by helping them make observations concerning the graphical information presented, by showing them alternative ways to display the information they have obtained or want to share, and by giving them practice in developing, to at least the conversational level, their skills in graphing.

Anyone working effectively within one of the STEM disciplines must necessarily have at least passing fluency in multiple categories of language. One category is that of natural languages, and English is (currently) the de facto standard worldwide for communication involving STEM. Mathematics is another category, and most of us can get by in many situations if we have a basic fluency in arithmetic, algebra, and elementary calculus. A third category is that of artificial languages, with the plethora of computer languages and dialects being ready examples. A fourth, and very important, category is that of visual languages. This category is likewise very broad. Some basic examples of visual languages directly applicable to STEM might include: architectural drawing, mechanical drawing, schematic drawing in its various dialects, and two- and three-dimensional graphics for presenting quantitative information.

Having good facility with two-dimensional graphics as it relates to mathematical quantities and relationships proves most helpful in developing deeper understanding of the mathematics describing signals, systems, their models, and physical phenomena in general. For this paper, we will limit our scope to that of two-dimensional graphics. Even so, a wealth of topics and considerations exists, with most items requiring anywhere from one to dozens of pages to develop fully. Thus, the following will take more or less the form of an outline, with the hope being that those interested will take the kernels and grow the details for themselves as they see fit. The various items for consideration are organized loosely into groups, with each group placed under its own heading.

Preliminaries

To be able to sketch functions, quantities and relationships quickly, one needs to commit a few (approximate) values and a few mathematical properties and relationships to memory. And assumptions must be made as to what the student knows. For example, it would be reasonable to expect a STEM student at the collegiate level to be familiar with logarithms, basic trigonometry, elementary functions, and conic sections, although a bit of self-directed review might be in order for some.

A few preliminary items include:

- *Symmetry.* What are the definitions for even, odd, halfwave, and quarterwave symmetry of a function?
- *Complex numbers.* What are several views of a complex number? How does one convert quickly between rectangular and polar form? (This is a great opportunity to employ a simple graph.)
- *Some (approximate) numerical values to memorize.* STEM is replete with exponentials, sinusoidal functions, and exponentially decaying sinusoids. Thus, it is convenient to have in mind numerical values, to three decimal places, for each of the following: π; e, e^{-1}, e^{-2}, e^{-3}, e^{-4}; and $\cos\theta$ for $\theta = 0°, 30°, 45°, 60°$, and $90°$. Also, common logarithms are used often; knowing that $\log_{10} 2 \approx 0.3$ and $\log_{10} 3 \approx 0.477 \approx 0.48$ is about all that is really necessary for quick approximations. Finally, knowing that $\ln 2 \approx 0.69$ can save time on occasion, even though that value can be obtained easily from the other values memorized.
- *Some identities to memorize.* Two very important trigonometric identities to keep in mind are
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
and the corresponding identity for $\sin(\alpha + \beta)$. Several other very useful trigonometric relationships can be obtained quickly from these two.

Sketching a Graph

Getting students started with graphing as soon as practicable is important. What they do with their hands has a mysterious, but strong, connection with what happens in their heads. Plus, it is important to give them ample chance to practice so they develop both a useful skill and a useful habit in sketching. I start students out with a few basic “rules” to follow when sketching a graph. They are:

- Make the graph as self-contained as practicable.
- Choose an appropriate set of axes for the data being presented.
- Label each axis with (1) the variable whose values are being plotted and (2) the units used.

- Use a straightedge to draw the axes and any “straight” portions of curves.
- Use a french curve or a drafting spline to draw other-than-straight portions of curves.
- Scale the axes such that the graph is easy to interpret (e.g., scale such that the spacing between adjacent tics on a linear plot is [1, 2, or 5] $\times 10^{\text{integer power}}$).
- Give the graph a descriptive title.
- If more than one curve is placed on the same graph, distinguish them by line-type, and include a legend.
- Clearly mark all plotted data points that correspond to measurements taken. Use a relatively large, easily distinguished symbol (and not a small dot!) to locate each data point.
- There are various approaches, each with its own reasoning and merit, to “filling in” the curve between data points. The most common are:

  a. Draw a curve approximated by the data points. (Most persons, with a bit of practice, are pretty good at eyeballing an approximately least-squares fit to a set of data.)
  b. Connect pairs of adjacent points with straight-line segments (i.e., perform piecewise-linear interpolation).
  c. Connect the data points via some more-involved interpolation scheme such a polynomial interpolation or cubic-spline interpolation.

  In order of decreasing preference, my choice is (a), then (b), then (c), if the data points correspond to measurements that (inevitably) include an error component.
- Sign and date each graph placed in a lab notebook.

There should be no hesitation in sharing some of the finer, but extremely important, points in creating a graph, e.g., employing the proper use of “data ink.” Edward Tufte’s book *The Visual Display of Quantitative Information*[1] is one that everyone who creates a graph—even if it is just a sketch—should own and learn from. It, as well as his other three major books on presenting visual information, provides a wealth of good advice, good rules, and examples, both good and bad.

Tufte’s *sparklines*[2] are, in his words, “intense, simple, word-sized graphics” that can have an important place in technical documents, but they can also be useful when sketching ideas related to an analysis or a design. Once students have firmly in mind what information is important to include in a graph, it is quite straightforward to slide over to sparklines.

It is important, too, to introduce students to some of the basic hand tools useful to sketching graphs. From my experience, very few even knew what a french curve was when I first mentioned it. So I talk about writing instruments (pencils and pens), and erasers. I require students to carry some sort of straightedge with them, and I recommend that they own at least one french curve or a drafting spline. Engineers' and architects' drafting scales, templates (circle, symbol, etc.), eraser shields, drafting tape, compasses, protractors, other drafting instruments such as dividers—even drafting machines—receive some attention from me. At the end of the discussion, the students know what few tools they can get by with and still do a decent job, but my intent is that they become aware of the broader range of basic tools that can assist in creating a drawing.

Paper also receives some attention from me. Both the type of ruling and the quality of the paper stock can be important. For work other than layout sketches for architectural or mechanical purposes, I specify 5×5 quadrille. Quad-ruled engineering paper, composition books, and bound lab notebooks are often of mediocre quality, both in terms of dimensional accuracy and in terms of quality of paper stock, but they are generally adequate for the purposes of sketching graphs, etc.

Scales and Axes

The proper choice of scale for an axis on a graph can enhance significantly the viewer's ability to understand the data or curves involved, and it can allow tremendous insight into the underlying process yielding the results being observed. The two most common scales are the linear and the logarithmic, but plenty of others, such as the gaussian scale, are available for use when most appropriate.

A linear axis is, of course, simple to set up on quadrille, so nothing further need be mentioned about its construction. A logarithmic axis is not much more difficult. If we let one decade on a logarithmically scaled axis correspond to, say, a four-inch interval on 5×5 quadrille, and if we can remember that $\log_{10} 2 \approx 0.3$, then it is straightforward to establish, within that decade (which we will assume for purposes of discussion to be from 1 to 10), tic marks for 2, 2.5, 4, 5, and 8; and it turns out that each of those tics lies on one of the quadrille's ruled lines. Further, if we can also remember that $\log_{10} 3 \approx 0.48$, then we can also quickly set tics for 3, 6, and 9. If we need to mark the position of 7, we could just take the midpoint between 6 and 8, which would be a slightly low estimate. Or we could mark $\sqrt{50}$ as an approximation, but it would be a bit high. Finally, we could average the two previous estimates and hit the correct position for 7 to within about 0.005%!

It is also quick to mark useful points such as $\sqrt{2}$. And although we might not need $\sqrt{10}$ often, it is handy to note that $\pi \approx \sqrt{10}$, with the error on the logarithmic axis being less than 0.6 percent of the distance used to represent a decade (i.e., about 0.022 in for a 4-in decade). So mark π at the midpoint of the interval representing a decade, and mark 2π at 8/10 the distance along the decade. If the axis were, say, the frequency axis on a Bode plot, then marking it in rad/s on one side and in Hz on the other is quick and simple, and it allows for perfectly easy conversion between the two units.

Once it is evident how to generate quickly both linear and logarithmic axes, a student can then learn what common functional relationships look like when graphed on various sets of axes. Plotting $y = ax + b$, $y = A\log_{10}\beta x$ (or, similarly, $y = A\ln\beta x$), $y = A\exp(\beta x)$, $y = x^\gamma$, and $y = 1/x^\gamma$ on linear–linear, linear–log, log–linear, and log–log sets of axes can be very insightful to someone who has never done that before.

Finally, a quick look at generating the axes for a polar plot is justified. The mechanics are very straightforward.

Sketching Simple Functions

A few simple functions are of particular importance to science and engineering, principally because they appear in so many different settings. They include: the Dirac delta (or, impulse) function $\delta(t)$, the Heaviside unit-step function $u(t)$, the decaying exponential $\exp(-t/\tau)$, the cosinusoidal function $\cos(\omega t+\phi)$, the exponentially decaying cosinusoidal function $\exp(-t/\tau)\cos(\omega t+\phi)$, and $\operatorname{sinc} t$.

The Dirac delta is not a function in the ordinary sense, but we must be able to represent it graphically. During discussion of what the graphical representation of $\delta(t)$ is trying to convey, the time is ripe for introducing students to the concept of a distribution function, of which $\delta(t)$ is an example, so they at least know why a mathematician would laugh violently at the typical definition given by engineers. Unit steps are easy to draw such that a value at any t can be read from the graph with great accuracy. And with the approximate values mentioned under *Preliminaries* in mind, one can sketch exponentials and cosinusoids, not only quickly but with enough precision that the value of the function at any t can be read from the graph to within better than 5%. The decaying exponential and the sinc function can be sketched quickly if only the values and locations of the relative extrema, the locations of the zeros, and perhaps the values of a few other points are all that need to be plotted accurately.

Further Techniques, Graphical Analysis, Applications, and Additional Topics

Familiarity and facility with all the foregoing make it straightforward to address a wide range of techniques, approaches and applications that can impart greater understanding when investigating, analyzing, designing or documenting a phenomenon, process or system. Some of the items on the list could be considered universal to all the STEM disciplines, while others might favor a select few. However, it is good to create *some* sort of a list because the list can help stimulate thought of other items to add. Each item might take a day or a week or two-semester sequence of instruction to develop, so for our present purposes, a simple list will suffice. Some topics, with a bias toward physics, electrical and computer engineering, and mechanical engineering, might include:

- Integration and differentiation: review of concepts; introduction to graphical approaches.

- Linearity: definitions; some important consequences of linearity; summing functions graphically; graphical convolution.
- Side-by-side time-domain and frequency-domain views of signals and systems.
- Nonlinear equations: overview of approaches to solution; solution by graphical means; application examples (establishing operating points, predicting harmonic distortion from characteristic curves, etc.).
- First- and second-order systems: definitions; basic characteristics; various views in the time and frequency domains.
- Fourier series and Fourier transforms: definitions; basic properties; spectra and their properties; correlation; graphical correlation.
- Pole–zero diagrams.
- Frequency response, asymptotic plots, Bode diagrams and their construction, and introductory concepts regarding stability.
- Nyquist diagrams, and introductory concepts of conformal mapping.
- Root-locus diagrams. How relating the underlying mathematics and their graphical consequences can lead to straightforward rules for constructing graphical views, and, more importantly, insight into the system being studied. (Side-topic: the spirule as an example of a clever mechanical aid to computation.)
- Bilinear transformations: definition; some basic applications. The Smith chart as an example, and introduction to its use.
- Graphically aided estimation and approximation. A few examples include visual estimation of: least-squares curve fitting, derivatives, integrals, and bandwidth (based on time-domain views). See Mahajan's *Street-Fighting Mathematics*[3] for a delightful exposure to the art of approximation.

To describe a concept together with graphical construction, graphical analysis, interpretation, and examples of applications pertinent to the concept can prove to be a very powerful pedagogical approach, and it can set up a path to deep appreciation and understanding. Furthermore, such integrated discussions have straightforward segues into introductions to numerical methods that can be applied to extend insight and to increase precision as the analysis of a problem proceeds to a deeper level.

Closure

While most of them are out of print, good books have been written that cover the construction of graphs and basic graphical analysis (see, e.g., Stein[4]); construction of graphical calculators, including nomographs (see Crowhurst[5]); and graphical techniques for addressing particular, but broad, classes of applications (see, e.g., Evans' text[6] covering his root-locus technique). Excellent chapters are written as introductions to graphical analysis, one such being found in Janert's *Gnuplot in Action.*[7] Gnuplot, a plotting package that is copyrighted but freely distributed, runs on several platforms, is capable of producing graphs of true publication quality, and makes a welcome tool for the "next step" past manual techniques in thinking via pictures.

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Biography

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